Causal Analysis for Both Tails in Time Series: With Application to China's Derivatives Market

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Introduction

- **Motivation**: existing literature on causality in multivariate extreme value theory tends to overlook the interaction between the upper and lower tails of random variables.
- **Contribution**: 1) we propose a new method to detect causal relationships for both tails of time series. 2) we contribute a high-frequency dataset from China's derivatives market.

Regular variation in a transformed space

- Regular variation $\mathbf{X} \in RV_+^p(\alpha)$ is often assumed in multivariate extreme statistics; $\mathbf{X}$ has tail decay according to a power-law with index α and an angular measure $H_{\mathbf{X}}(\cdot)$ on the unit sphere $\mathbb{S}_{p-1}^+ = \{\mathbf{x} \in \mathbb{R}_+^p : ||\mathbf{x}||_2 = 1\}$.
- With $t(x) = \log(\exp(x)+1)$, the space $\mathbb{X}^p = \{t(\mathbf{x}) : \mathbf{x} \in \mathbb{R}^p\}$ is an inner product space (Cooley & Thibaud, 2019) with transformed linear operations (Table 1).

Operation	Definition
addition	$x_1 \oplus x_2 = t(t^{-1}(x_1) + t^{-1}(x_2))$
scalar multiply	$a \circ x = t(at^{-1}(x))$
inner product	$\langle x_1, x_2 \rangle = \sum_{i=1}^p t^{-1}(x_{1i})t^{-1}(x_{2i})$

Figure 1: Transformed operations

Regular variation is preserved in $\mathbb{X}^p$

- Tail Pairwise Dependence Matrix (TPDM) summarizes extremal dependence: $\Sigma_{\mathbf{X}} = \{\sigma_{ij}\}$, where $\sigma_{ij} = \int_{\mathbb{S}_{p-1}^+} w_i w_j dH_{\mathbf{X}}(\mathbf{w})$.
- Partial Tail Correlation Coefficient (PTCC):

$\gamma_{ij|\mathbf{X}_{-ij}} = \frac{\langle X_i \ominus \widehat{X_i}, X_j \ominus \widehat{X_j} \rangle}{||X_i \ominus \widehat{X_i}|| \, ||X_j \ominus \widehat{X_j}||}$; $\mathbf{X_{-ij}}$ is $\mathbf{X}$ without (X_i, X_j) and $\widehat{X_i}$ is the optimal transformed predictor for X_i given $\mathbf{X_{-ij}}$.

We **can test** for partial tail uncorrelatedness ($\gamma_{ij|\mathbf{X}_{-ij}} = 0$) denoted as $X_i \perp_p X_j | \mathbf{X_{-ij}}$, following (Lee & Cooley, 2022).

Transformed linear causal model

Let $X_i^u = T(X_i^{original})$, $X_i^d = T(-X_i^{original}) \in RV_+(2)$.

For $\mathbf{X}^{original} \in \mathbb{R}^p$, construct $\mathbf{X} = [X_1^u, X_1^d, \ldots]' \in RV_+^{2p}(2)$.

Inference for one tail

Structured Causal Model (SCM) represented by a Directed Acyclic Graph (DAG) needs 1) a graph G; 2) a causal equation system: $X_i^u := \eta_i^u \oplus \bigoplus_{X_j^u \in Pa(X_i^u)} X_i^u \circ \beta_{j\to i}^u$. The parameter space for this model is $\Theta_1 = (C, B)$, where C stores β and $B = diag(\sigma_{\eta_1}, \ldots)$.

Theorem 1 $\Sigma_{\mathbf{X^u}} = (I-C)^{-1}B[(1-C)^{-1}]'$ (system's TPDM)

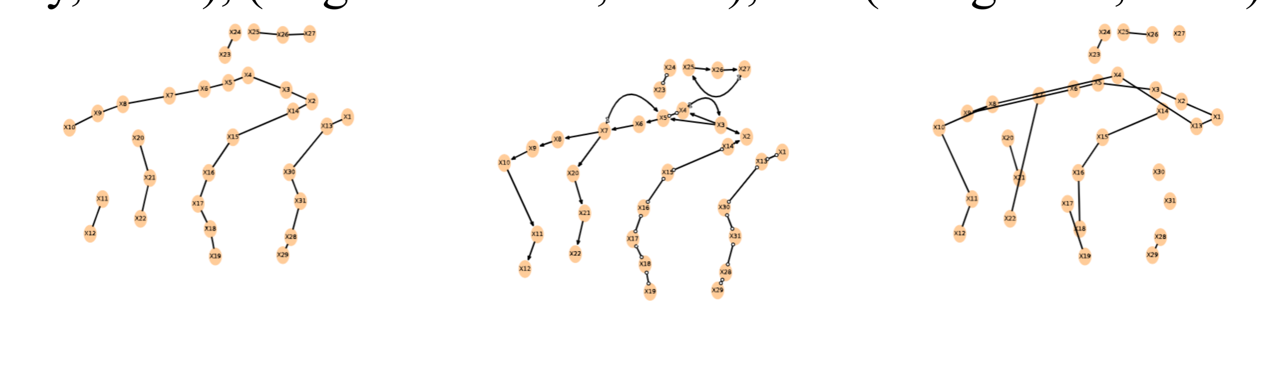
We propose learning the SCM by PCMCI $^+$ (Runge, 2020) and PTCC

Phase 1: Skeleton discovery - iteratively test $X_i \perp_p X_j | \mathbf{X_{-ij}}$

Phase 2: Edge orientation, which gives the causal direction

Application: Danube dataset

Danube data contains daily river discharges (n=428) from 31 stations. Here we estimate SCM and do comparison with: (Lee & Cooley, 2022), (Engelke & Hitz, 2020), and (Gong et al., 2022)



(a) Estimation in Gong et al. (2022) (b) Estimation in this study (c) Estimation in Lee and Cooley (2022)

(d) Estimation in Engelke and Hitz (2020) (e) Linear Causality (f) Tail causality on negative side

Figure 2: Comparison for Danube river discharge dataset; our model can show direction and temporal dependence.

Application: high-frequency China's derivatives market

New Dataset collection

- It is a thriving market (turnover is 87 trillion ￥CNY in 2021)
- Include orderbook (2 records/sec); 80 million records a day;
- Potential research topics: data mining, risk-contagions, market efficiency and microstructure, high-frequency trading.

code	name	exchange	**code**	name	exchange
rb	Rebar	SHFE	**b**	Soybean Type.2	DCE
m	Soybean Meal	CFFEX	**j**	Coke	DCE
a	Soybean Type.1	DCE	**jm**	Coking Coal	DCE
al	Aluminum	SHFE	**cu**	Copper	SHFE
CY	Cotton Yarn	CZCE	**y**	Soybean Oil	DCE
i	Iron Ore	CZCE	**zn**	Zinc	SHFE

Figure 3: Derivative products for China's derivatives market

Estimation of causal structure

- 21 products, 4 sectors (color metal, chemicals,...).
- Results show one-step (5 seconds) forward SCM (Figure 4)
- Most are symmetric, suggesting market's efficiency in tails

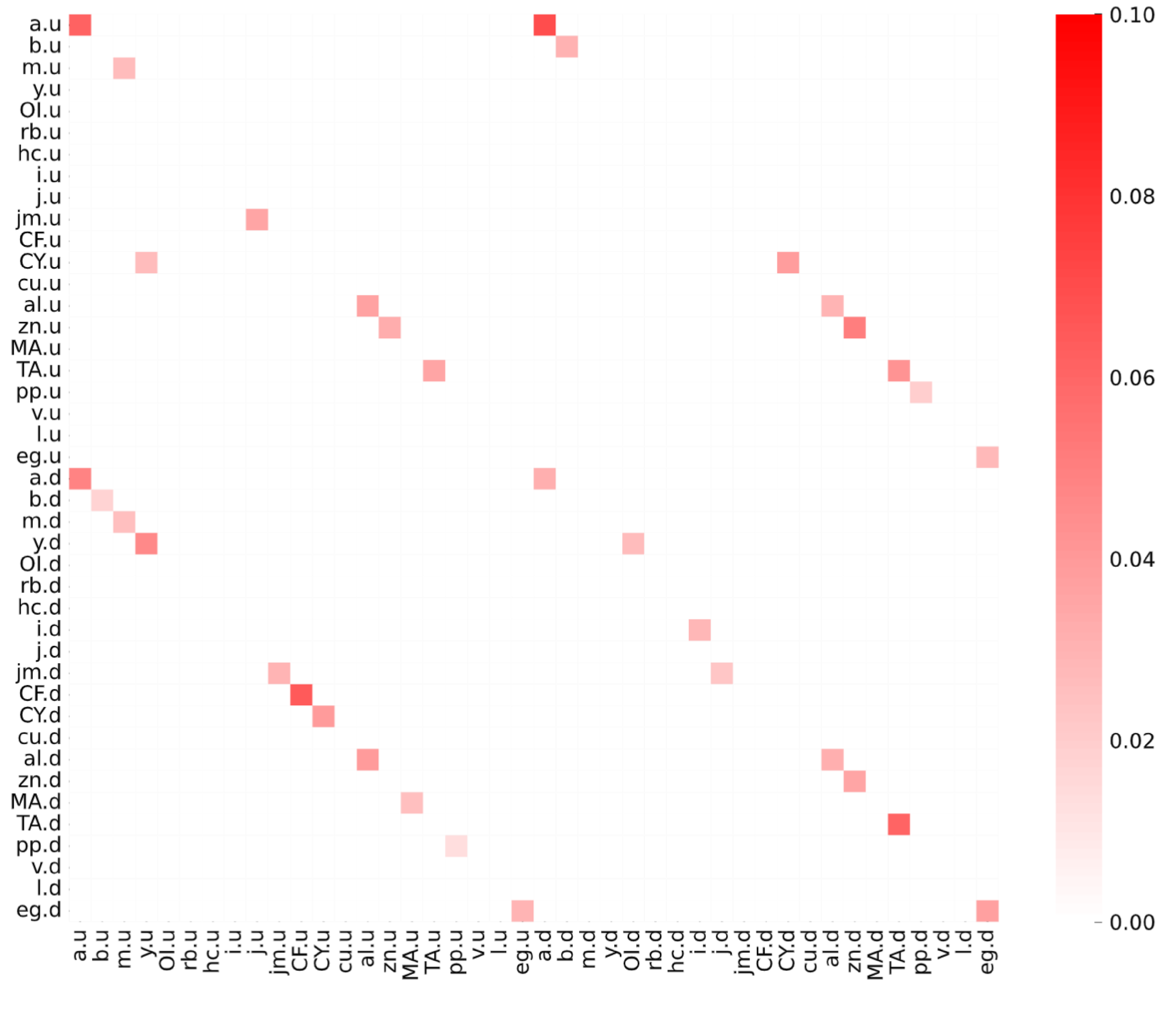
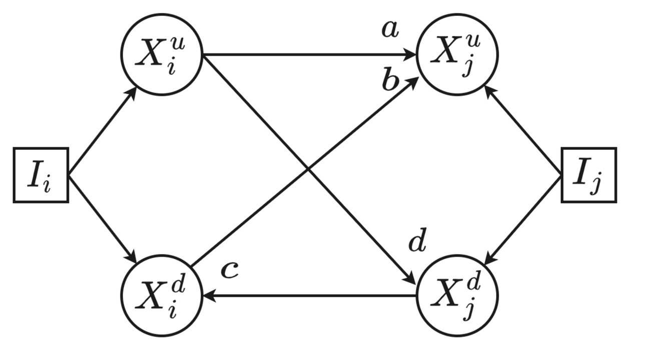


Figure 4: One-step forward causality for the China`s derivatives return; quantified by PTCC.

Transformed linear causal model for both tails

- Since the orthogonality constraint $\sigma_{(X_i^u, X_i^d)} = 0$ exists:
- $X_i^u := I_i[\eta_i^u \oplus \bigoplus_{X_j^e \in Pa(X_i^u)} X_i^e \circ \beta_{j\to i}^e]$;
  $X_i^d := (1-I_i)[\eta_i^d \oplus \bigoplus_{X_j^e \in Pa(X_i^d)} X_i^e \circ \beta_{j\to i}^e]$
- Swither $I_i = \mathbf{1}(X_i^{ori} > 0)$ are binary R.Vs (Illustrated in Figure 5).



$$\begin{matrix} X_i^u \\ X_i^d \\ X_j^u \\ X_j^d \end{matrix} \begin{bmatrix} 0 & 0 & a & d \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 \\ 0 & c & 0 & 0 \end{bmatrix}$$

Figure 5: Left: one example of a causal graph for generalized transformed linear causality. Right: Causal matrix C for those 4 variables.

Conclusion

- We propose a transformed causal model which can nicely model the causal structure for both tails of time series data.
- Most of the lead-lag dependence relationship are symmetric in China's derivatives market.
- Interaction between both tails should receive more attention.
- We build a high-frequency dataset for China's derivatives market.

References

Cooley, D., & Thibaud, E. (2019). Decompositions of dependence for high-dimensional extremes. *Biometrika*, *106*(3), 587–604.

Engelke, S., & Hitz, A. S. (2020). Graphical models for extremes. *Journal of the Royal Statistical Society Series B: Methodology*, *82*(4), 871–932.

Gong, Y., Zhong, P., Opitz, T., & Huser, R. (2022). Partial tail-correlation coefficient applied to extremal-network learning. *arXiv Preprint arXiv:2210.07351*.

Lee, J., & Cooley, D. (2022). Partial tail correlation for extremes. *arXiv Preprint arXiv:2210.02048*.

Runge, J. (2020). Discovering contemporaneous and lagged causal relations in autocorrelated nonlinear time series datasets. *Conference on Uncertainty in Artificial Intelligence*, 1388–1397.