A New Dependence Measure for Extremal Brain Connectivity

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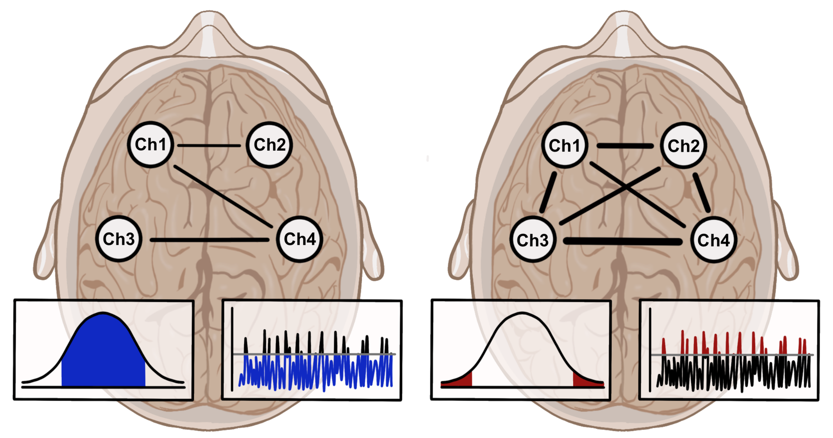
Abstract

- We develop a new conditional extremal dependence measure that captures the strength of relationship (beyond linear dependence) between two random variables in a tail region given other variables in a network.
  - **Advantages**: (i) model-free; (ii) computed in time $O(n \log n)$; (iii) inference using a nonparametric test.
  - **Motivation**: Difference between classical and extremal brain connectivity.



- **Caveat**: We define extremal dependence as the dependence between variables in a tail region which is not equivalent to asymptotic dependence, i.e., $\lim_{u\to 1} \mathbb{P}(Y > F_Y^{-1}(u) \mid X > F_X^{-1}(u)) > 0$.

Conditional Extremal Dependence

Let $(X, Y, Z)^\top$ be a random vector and consider a quantile level $u \in [0, 1]$. Define the new triple $(X^*, Y^*, Z^*)^\top$ as $(X, Y, Z)^\top \mid X > F_X^{-1}(u)$, where $F_X^{-1}(u)$ is the u-quantile of X. For a specified quantile level u, the conditional extremal dependence between X and Y given Z is measured by

$$\xi_u(X, Y \mid Z) = \frac{\int \mathbb{E}\left[\mathrm{Var}(\mathbb{P}(Y^* \geq t \mid X^*, Z^*) \mid Z^*)\right] \mathrm{d}\mu^*(t)}{\int \mathbb{E}\left[\mathrm{Var}(1_{\{Y^* \geq t\}} \mid Z^*)\right] \mathrm{d}\mu^*(t)} \in [0, 1],$$

where $\mu^*(\cdot)$ is the distribution of $Y^* = Y \mid X > F_X^{-1}(u)$ [extends [1]].

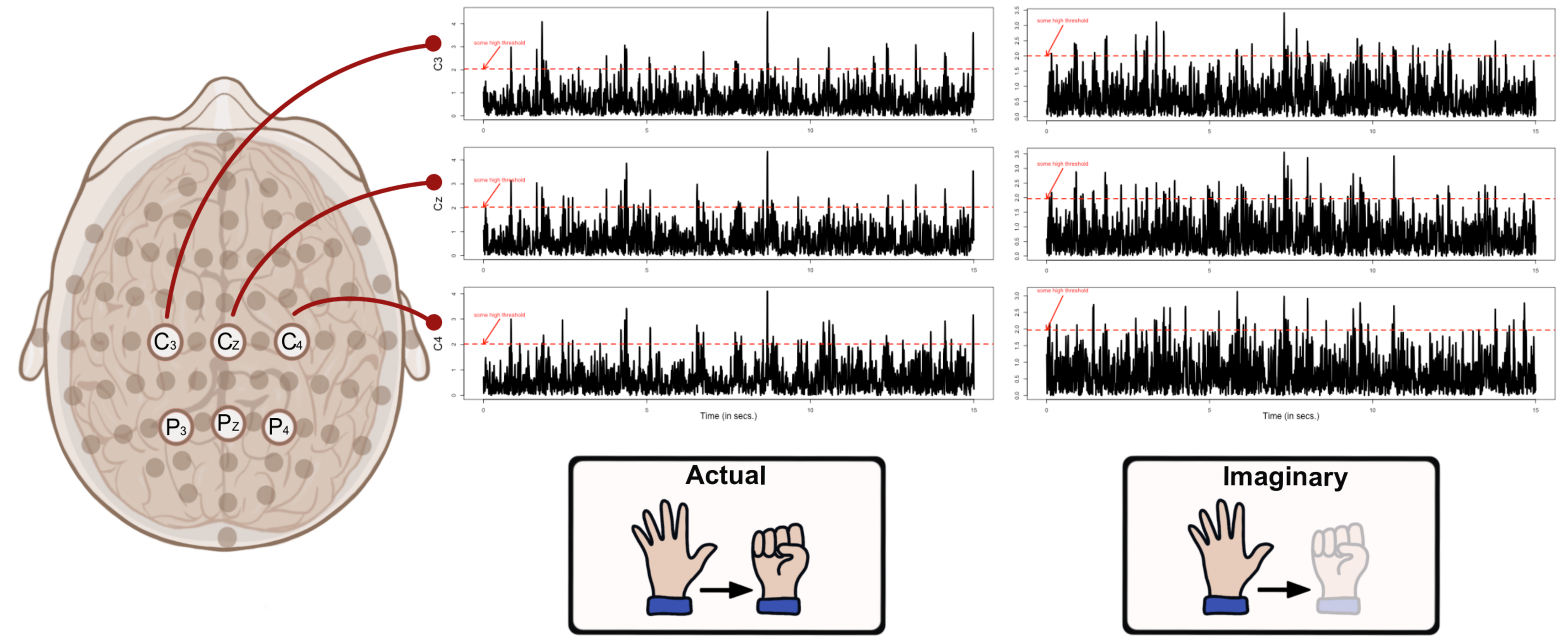
- $\xi_u(X, Y \mid Z) = 0$ if and only if X^* and Y^* are independent given Z^*, and $\xi_u(X, Y \mid Z) = 1$ if and only if Y^* is a function of X^* given Z^*.
- $\xi_u(X, Y \mid Z)$ is not symmetric.
- **Remark:** We say $(X^*, Y^*, Z^*)^\top$ is in the tail region C_u, which is defined by X exceeding a high threshold.

Motor Movement/Imagery EEG Analysis

Data: EEG recordings (at 160 Hz) during performance of motor-related tasks from ten volunteers.

Tasks: (i) actual movement (open/close fist), and (ii) imaginary action.

Selected Channels: C3, Cz and C4 – body sensory and motor functions; P3, Pz and P4 – facilitate accurate upper limb movements.



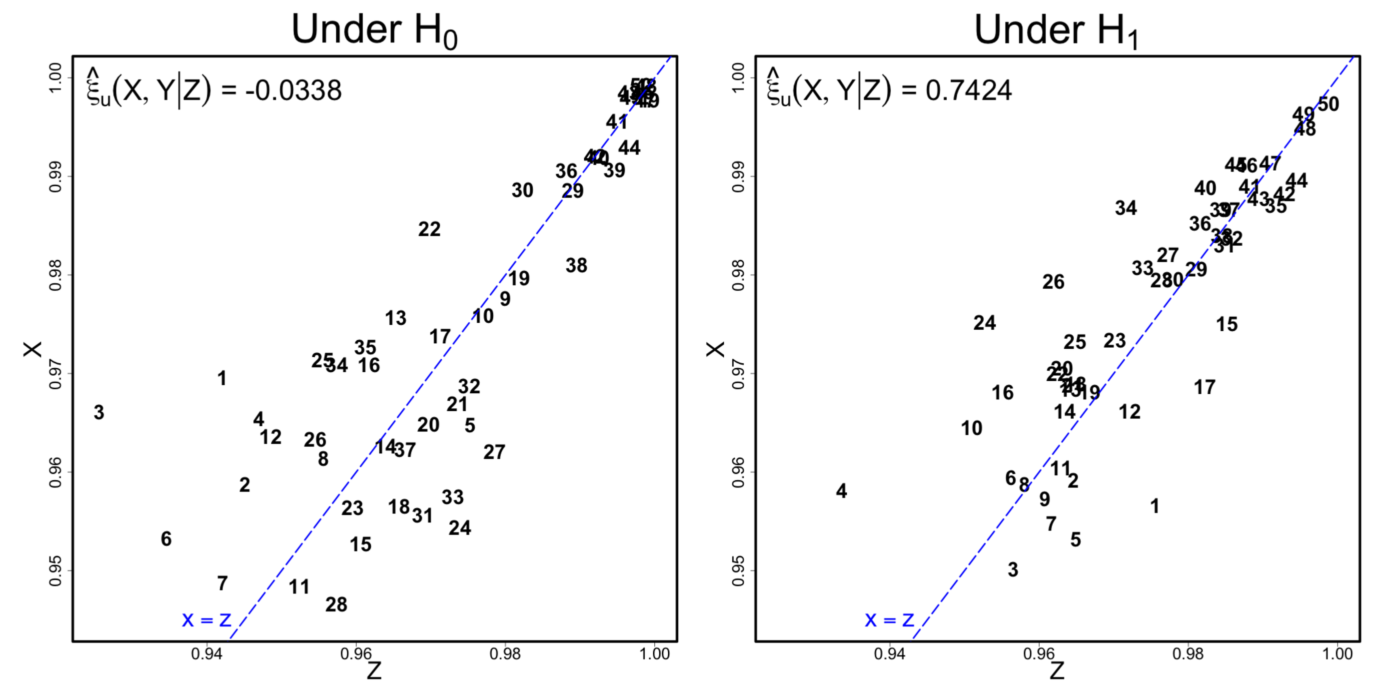
Nonparametric Inference on Conditional Extremal Dependence

Estimation: Suppose data consist of iid triples $(X_i, Y_i, Z_i)^\top, i = 1, \ldots, m$. For a quantile level u, define new obs. $(X_j^*, Y_j^*, Z_j^*)^\top$ such that $X_j^* > F_X^{-1}(u)$ for all $j = 1, \ldots, n$. The estimator of $\xi_u(X, Y \mid Z)$ is

$$\hat{\xi}_u(X, Y \mid Z) = \frac{\sum_{j=1}^{n} (\min\{R_j, R_{M(j)}\} - \min\{R_j, R_{N(j)}\})}{\sum_{j=1}^{n} (R_j - \min\{R_j, R_{N(j)}\})},$$

where R_j is the rank of Y_j^*, $N(j)$ is the index k such that Z_k^* is the nearest neighbor of Z_j^* on $\mathbb{R}$, and $M(j)$ is the index k such that $(X_k^*, Z_k^*)^\top$ is the nearest neighbor of $(X_j^*, Z_j^*)^\top$ in $\mathbb{R}^2$.

Behavior of Ranks R_j:



Test for Independence in the Tail Region C_u:

Suppose we wish to test the null hypothesis that X and Y are independent in the tail region C_u given Z, i.e.,

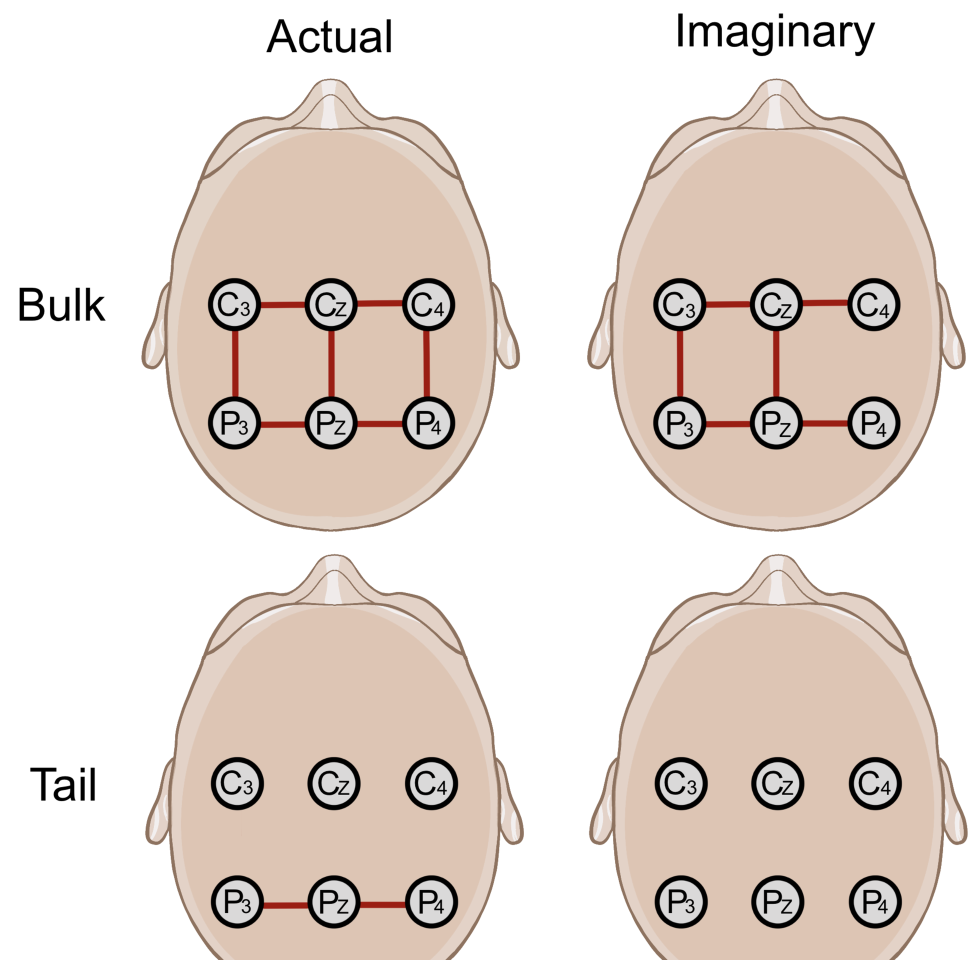
$$H_0 : \xi_u(X, Y \mid Z) = 0 \quad \text{vs.} \quad H_1 : \xi_u(X, Y \mid Z) \neq 0.$$

We construct the empirical null distribution of the estimator $\hat{\xi}_u(X, Y \mid Z)$ by simple "reshuffling" of adjacent ranks.

Outline:

1. Generate independent random permutations of $R_{M(j)}$ and $R_{N(j)}$.
2. Calculate $\hat{\xi}_u(X, Y \mid Z)$ based on the reshuffled ranks and denote by $\hat{\xi}_u^{(b)}(X, Y \mid Z)$.
3. Compute p-value as $|\{\hat{\xi}_u^{(b)}(X, Y \mid Z) \geq \hat{\xi}_u(X, Y \mid Z)\}|/B$, where B is the number of reshuffles considered.
4. Given significance $\alpha \in (0, 1)$, reject the null hypothesis if the p-value is less than α. Otherwise, do not reject H_0.

Estimated Brain Connectivity Network:



Novel Findings:

- There is a direct link (represented by a red line) between two channels when their signals are conditionally dependent given all other channels in the network
- **Bulk**: Similar systematic brain connectivity for the two tasks.
- **Tail**: Prominent direct links in the parietal region when performing actual movement while conditional independence during imaginary action.

Future Work:

- Develop a causal extremal measure, e.g., $\xi_u(\boldsymbol{X}_{t-k}, Y_t \mid \boldsymbol{Y}_{t-\ell})$.

References:
[1] Azadkia, M. and Chatterjee, S. *simple measure of conditional dependence.* The Annals of Statistics 49 (2021): 3070–3102.
[2] Chung, J. W., et al. *Beta-band activity and connectivity in sensorimotor and parietal cortex are important for accurate motor performance.* Neuroimage 144 (2017): 164–173.

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