Extreme Causal Analysis for Both Tails in Time Series Data

The Efficient Tail Hypothesis

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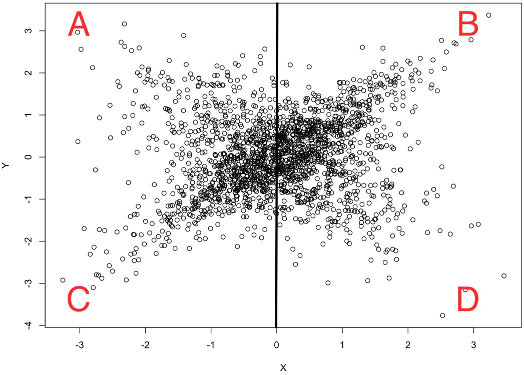
Motivation

- The Efficient Market Hypothesis states that the expected value of an asset return, given measurable information from the past, is zero;
- Efficiency drops under extreme conditions (Choi, 2021);
- We study **asymmetry** in bivariate tail dependence via a new dependence measure: directional-tail-dependence λ;
- To construct the *Efficient Tail Hypothesis*.

Motivation



- X returns of some asset at time $t-1$: Y target asset at time t.
- $A = B$: Tail symmetric; $A = D$: directional tail asymmetric.

Multivariate regular variation

Definition (Multivariate regular variation (§6, Resnick (2007)))

A p-dimensional random vector $\boldsymbol{X} \in \mathbb{R}_+$ is regularly varying (RV) with tail index $\alpha > 0$, denoted by $\boldsymbol{X} \in \mathrm{RV}_+^p(\alpha)$, if there exists a sequence $b_n \to \infty$ such that $n\mathbb{P}(b_n^{-1}\boldsymbol{X} \in \cdot) \xrightarrow{v} \nu_{\boldsymbol{X}}(\cdot)$ as $n \to \infty$.

- Radius $R := \|\boldsymbol{X}\|_2$ and angles $\boldsymbol{X}/R$ independent (in limit);
- Radial–angular decomposition of limit measure $v_{\boldsymbol{X}}(\cdot)$:

$$\nu_{\boldsymbol{X}}(\{\boldsymbol{x} \in \mathbb{E}_+^p : ||\boldsymbol{x}||_2 \geq r, \boldsymbol{x}/||\boldsymbol{x}||_2 \in \mathcal{B}\}) = r^{-\alpha} H_{\boldsymbol{X}}(\mathcal{B}),$$

  for cone $\mathbb{E}_+^p = [0, \infty]^p \setminus \{\mathbf{0}\}$ and $\mathcal{B} \subset \mathbb{S}_{p-1}^+$;
- Angular mass measure $H_{\boldsymbol{X}}(\cdot)$ is defined on the positive part of unit sphere $\mathbb{S}_{p-1}^+ = \{\boldsymbol{x} \in \mathbb{R}_+^p : ||\boldsymbol{x}||_2 = 1\}$;
- $H_{\boldsymbol{X}}(\cdot)$ is ID card for upper tails of $\boldsymbol{X}$ (**but difficult to estimate**).

- Extremal dependence measure (EDM) for $(X, Y)^\top \in \mathrm{RV}_+^2(2)$:

$$\sigma(X,Y) := \int_{\mathbb{S}_+^1} \omega_x \omega_y \mathrm{d}N_{(X,Y)}(\boldsymbol{\omega}) = \lim_{r\to\infty} \mathbb{E}\left[\frac{XY}{R^2} \mid R > r\right], \qquad (1)$$

  where $R = ||(X,Y)||_2$ and $N_{\boldsymbol{X}}(\cdot) := \frac{H_{\boldsymbol{X}}(\cdot)}{H_{\boldsymbol{X}}(\mathbb{S}_{p-1}^+)}$ is a probability measure;
- $\sigma(X, Y) = 0 \Rightarrow$ tail independence;
- For $\alpha = 2$, $\sigma(X, Y) = 0.5 \Rightarrow$ perfect tail dependence.

- Regular variation is a popular modelling framework for multivariate extremes, but it is only defined on the positive orthant;
- $\Rightarrow$ Only facilitates learning of joint extremal behaviour;
- Defining regular variation on the entire reals is possible:

Methodology

Regular variation on $\mathbb{R}^p$

Definition (Regular variation on $\mathbb{R}^p$)

A p-dimensional random vector $\boldsymbol{X} \in \mathbb{R}^p$ is regularly varying with tail index $\alpha > 0$, denoted by $\boldsymbol{X} \in \mathrm{RV}^p(\alpha)$, if $|\boldsymbol{X}| \in \mathrm{RV}_+^p(\alpha)$ with the normalizing sequence $b_n \to \infty$ and for all $\boldsymbol{z} \in [0, \infty)^p \setminus \{\boldsymbol{0}\}$ and $\boldsymbol{s} \in \{-1, 1\}^p$,

$$\lim_{n\to\infty} \frac{\mathbb{P}\{b_n^{-1}\boldsymbol{s} \odot \boldsymbol{X} \in [\boldsymbol{0}, \boldsymbol{z}]^c\}}{\mathbb{P}\{b_n^{-1}|\boldsymbol{X}| \in [\boldsymbol{0}, \boldsymbol{z}]^c\}} \in (0, 1),$$

where $\odot$ is the element-wise (Hadamard) product and $|\cdot|$ denotes the element-wise absolute value.

- Has limit and angular measures, $\nu_{\boldsymbol{X}}(\cdot)$ and $H_{\boldsymbol{X}}(\cdot)$.
- $H_{\boldsymbol{X}}(\cdot)$ defined on $\mathbb{S}_{p-1} = \{\boldsymbol{x} \in \mathbb{R}^p : ||\boldsymbol{x}||_2 = 1\}$.

Methodology

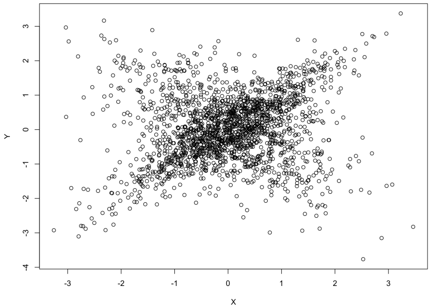
Balanced regular variation on $\mathbb{R}^p$

- Working with $\mathrm{RV}^p(\alpha)$ variables can be troublesome as there are no constraints on how the mass of $H_{\boldsymbol{X}}(\cdot)$ is distributed across $\mathbb{S}_{p-1}$.
- One trick is to use balanced regularly varying random variables $\mathrm{BRV}^p(\alpha)$ with the following marginal constraint:

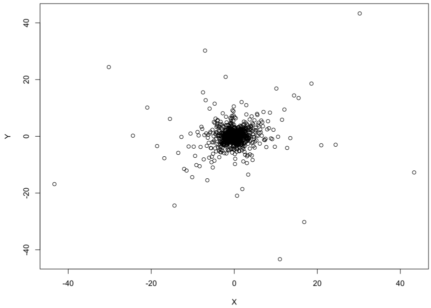
Definition (Balanced regular variation)

A p-dimensional regularly varying random vector $\boldsymbol{X}$ is balanced, denoted by $\boldsymbol{X} \in \mathrm{BRV}^p(\alpha)$, if $\boldsymbol{X} \in \mathrm{RV}^p(\alpha)$ and $v_{\boldsymbol{X}}(\{\boldsymbol{x} \in \mathbb{E}^p : x_i > 1\}) = v_{\boldsymbol{X}}(\{\boldsymbol{x} \in \mathbb{E}^p : x_i < -1\}) = 1$ for all $i = 1 \ldots, p$.

- WLOG as known transformation.



Y
X
-3
-2
-1
0
1
2
3
-4



Y
X
-40
-20
0
20
40

- We now constrain our focus to bivariate vector $(X, Y)^\top \in \mathrm{BRV}^2(2)$.
  - X: explanatory variable, typically lagged $(t-1)$ asset price;
  - Y: target variable, asset price at time t ;
- A useful trick (in financial extremes) is truncation at zero. Let $Y^+ := \max\{Y, 0\}$ and $Y^- := -\min\{Y, 0\}$.
- Studying directional asymmetry in $(X, Y)^\top$ is equivalent to quantifying differences between the strength of **positive tail dependence** in $(X, Y^+)^\top$ and $(X, Y^-)^\top$.

Methodology

Extremal ball for visualizing directional tail dependence

- To visualise asymmetry, evaluate $\sigma(X, Y^+)$ and $\sigma(X, Y^-)$.
- Project EDM to the *extremal ball* (positive part of $\mathbb{S}_2^+$).

Methodology

Extremal ball for visualizing directional tail dependence

Definition (Extremal ball)

For $(X, Y)^\top \in \mathrm{BRV}^2(2)$ with Y^+, Y^- as before, define coordinates $(\theta_{X,Y^+}, \theta_{X,Y^-})$, where

$$\theta_{X,Y^+} = \cos^{-1} \frac{\sigma(X, Y^+)}{\sqrt{\sigma(X, X)\sigma(Y^+, Y^+)}}, \tag{2}$$

denotes the angle between X and Y^+.

Can similarly get θ_{X,Y^-}.

Methodology

Relative position of X^+ to Y on extremal ball

- Points on the tail efficient line imply symmetric dependence;
- Deviation from the central line implies asymmetric dependence.

Figure: Extremal ball.

Empirical evidence

Dataset

- Percentage price change (interval is 30 seconds) of futures of commodities in China,
- Contains 55 different assets across different sectors
- From 2022-08-01 to 2023-07-31 and contains 110985 observation,
- Transformed data: $\boldsymbol{Z}_t = (Z_{1,t}, \ldots, Z_{55,t})^\top$ for $t = 1, \ldots, T$.

Empirical evidence

Market-wide directional tail symmetry analysis

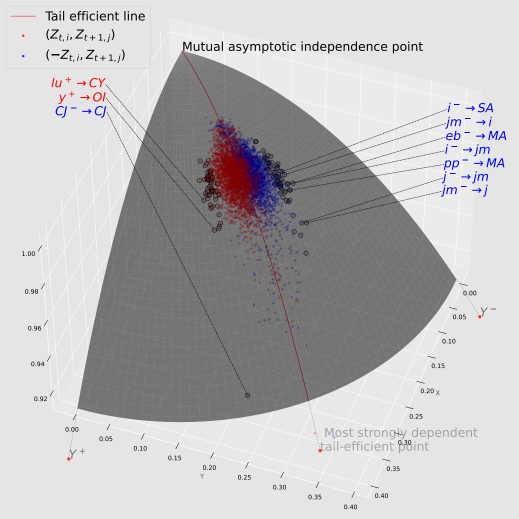
Extremal ball:

- visualize $(Z_{i,t}, Z_{j,t+1})^\top$ and $(-Z_{i,t}, Z_{j,t+1})^\top$, $i, j = 1, \ldots, 55$.
- in total 6050 points
- The former: the driven event is an extremal appreciation,
- The latter: the driven event is an extremal depreciation,
- most points concentrate on the tail efficient line.

Empirical evidence

Market-wide directional tail symmetry analysis

- Points on the tail efficient line imply symmetric dependence;
- Deviation from the central line implies asymmetric dependence.



Methodology

Directional tail dependence λ

Definition (Directional tail dependence measure)

For $(X, Y)^\top \in \mathrm{BRV}^2(2)$, the directional tail dependence measure, $\lambda(X, Y)$, is defined by

$$\lambda(X, Y) = \int_{\{(\omega_x, \omega_y)^\top \in \mathbb{S}^1 : \omega_x \geq 0\}} \omega_x \omega_y \mathrm{d}H_{(X,Y)}(\boldsymbol{\omega}). \tag{3}$$

Properties of λ

- $\lambda(X, Y) \in [-1, 1]$, boundary case ± 1 are achieved when $\lim_{a\to\infty} \mathbb{P}[X > a \mid Y < -a] = 1$ or $\lim_{a\to\infty} \mathbb{P}[X > a \mid Y > a] = 1$;
- $\lambda(X, Y) = 0$ implies directional symmetry (tail efficient);
- **Estimation of λ is not straightforward** as we don't know the angular mass on the right half circle, $H_{(X,Y)}(\{(\omega_x, \omega_y)^\top \in \mathbb{S}^1 : \omega_x \geq 0\})$.

Methodology

Estimation for λ

Proposition (Two equivalent form of $\lambda(X, Y)$)

$$\begin{aligned}\lambda(X, Y) &= 3\int_{\{(\omega_x,\omega_y)^\top\in\mathbb{S}^1:\omega_x\geq 0\}} \omega_x\omega_y \mathrm{d}N_{(X^+,Y)}(\boldsymbol{\omega}) := \lambda^1(X, Y) \\ &= 2\{\sigma(X^+, Y^+) - \sigma(X^+, Y^-)\} := \lambda^2(X, Y).\end{aligned} \tag{4}$$

Definition (Two estimators for λ)

Let $\{(x_i, y_i)^\top\}_{i=1}^n$ draw i.i.d. from $(X, Y)^\top$.

1) $\widehat{\lambda}_n^1 = 3\frac{1}{k}\sum_{i=1}^n \frac{x_i^+ y_i}{r_i^2}\mathbb{1}(r_i \geq r_0)$

2) $\widehat{\lambda}_n^2 = 2\left(\frac{1}{k^+}\sum_{i=1}^n \frac{x_i^+ y_i^+}{(r_i^+)^2}\mathbb{1}(r_i^+ \geq r_0^+) - \frac{1}{k^-}\sum_{i=1}^n \frac{x_i^+ y_i^-}{(r_i^-)^2}\mathbb{1}(r_i^- \geq r_0^-)\right)$

where r_i, r_i^+, r_i^- is radius; e.g., $r_i = ||(x^+, y)^\top||_2$,
k, k^+, k^- are number of exceedences; e.g., $k^+ = \sum_{i=1}^n \mathbb{1}(r_i^+ > r_0^+)$
and r_0, r_0^+ and r_0^- are suitably-chosen high thresholds.

- We prove asymptotic normality for λ_n^1 and also construct a **non-parametric permutation test (for both).**

Methodology

Permutation test for ETH

- Under the null hypothesis, with $\lambda(X, Y) = \lambda(X, -Y) = 0$, we can reflect realisations of Y in the x-axis (with probability equal to 0.5) to produce new samples
- Works for both estimators $\widehat{\lambda}_n^1$ and $\widehat{\lambda}_n^2$.
- In simulation study (not discussed), we find that permutation test for $\widehat{\lambda}_n^2$ gives the strongest power.

Methodology

Efficient Tail Hypothesis (ETH)

Definition (pairwise Efficient Tail Hypothesis)

For $\left(\boldsymbol{X}^\top, \boldsymbol{Y}^\top\right)^\top \in \mathrm{BRV}^{p_1+p_2}(2)$, the Efficient Tail Hypothesis states that all pairs $(X_i, Y_j)^\top$, $i = 1, \ldots, p_1$, $j = 1, \ldots, p_2$, are tail-efficient, that is, $\lambda(X_i, Y_j) = 0$

- $\boldsymbol{Y} \in \mathbb{R}^{p_2}$ can be asset returns at time t
- $\boldsymbol{X} \in \mathbb{R}^{p_1}$ can be information that is measurable before time t

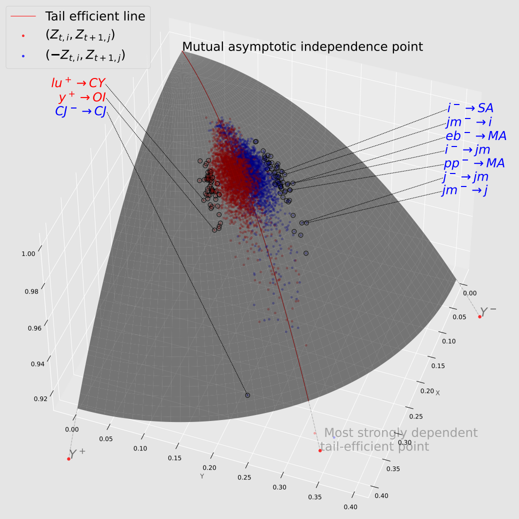
Test the ETH:

- Is the deviation significant?
- We let $\boldsymbol{X} = (Z_{1,t}, \ldots, Z_{55,t}, -Z_{1,t}, \ldots, -Z_{55,t})^\top$ be the explanatory vector and $\boldsymbol{Y} = (Z_{1,t+1}, \ldots, Z_{55,t+1})^\top$ be the target vector.
- using permutation test with 6050 individual tests,
- use multiple test correction (Benjamini and Hochberg, 1995),
- significance level $\alpha = 0.01$, the quantile for thresholds r_0^+, r_0^- are derived using the empirical 0.99-quantile.
- Null **rejected** with 126 significant pairs.

Empirical evidence

Market-wide directional tail symmetry analysis

126 tail inefficient pairs are found.
May present **profitable investment** opportunities.



Empirical evidence

Market-wide directional tail symmetry analysis

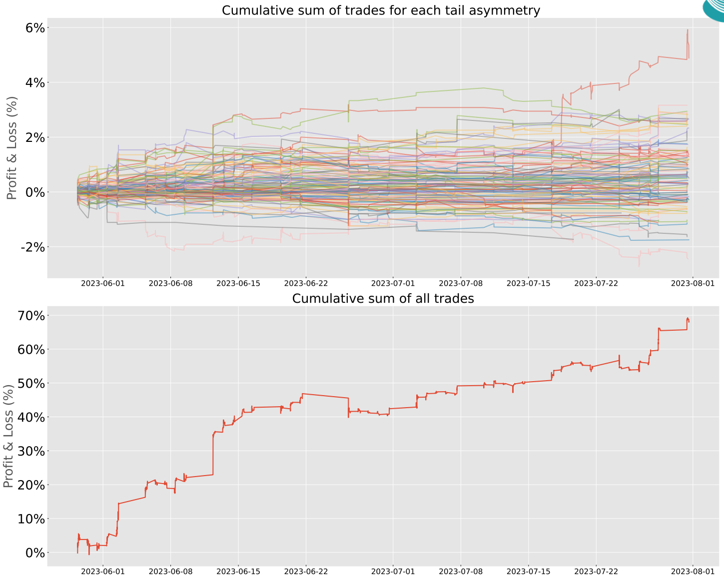
Table: Asset pairs (and their acronyms) with the top 10 largest $|\widehat{\lambda}_n^2(X_i, Y_j)|$.

Explanatory asset X_i	Target asset Y_j	Sign of X_i	$\widehat{\lambda}_n^2(X_i, Y_j)$
red dates (CJ)	red dates (CJ)	negative	0.073
coking coal (jm)	coke (j)	negative	-0.065
iron ore (i)	coking coal (jm)	negative	-0.063
soybean oil (y)	vegetable oil (OI)	positive	0.057
low sulfur fuel (lu)	cotton yarn (CY)	positive	0.056
coke (j)	coking coal (jm)	negative	-0.055
coking coal (jm)	iron ore (i)	negative	-0.055
styrene (eb)	methanol (MA)	negative	-0.055
polypropylene (pp)	methanol (MA)	negative	-0.054
iron ore (i)	soda ash (SA)	negative	-0.053

Empirical evidence

Artificial dynamic portfolio based on the significant tail-inefficient pair

- we construct an artificial dynamic portfolio based on the trading strategy on each of those 126 significant tail-inefficient pairs,
- for an inefficient pairs (X_i, Y_j):
  - buy/sell asset of Y_j when observing an extreme value for asset X_i;
  - For X_i, 99.5% historical quantile for positive extremes and falls below 0.5% the quantile for negative extremes;
  - buy if $\lambda(X_i, Y_j) > 0$, otherwise sell;
  - then close the position after one period;
  - trading cost is ignored;
  - we backtest this strategy on the out-of-sample data from 2023-06-01 to 2023-07-31.


Cumulative sum of trades for each tail asymmetry
Profit & Loss (%)
6%
4%
2%
0%
-2%
2023-06-01
2023-06-08
2023-06-15
2023-06-22
2023-07-01
2023-07-08
2023-07-15
2023-07-22
2023-08-01
Cumulative sum of all trades
70%
60%
50%
40%
30%
20%
10%
0%

Summary

KAUST

- We proposed a new measure, the directional tail dependence $\lambda(X, Y)$, to quantify the extent of directional tail symmetry;
- Empirically, the ETH is rejected for China's futures market;
- We use the 126 signifcant pairs to construct a profitable investment strategy;
- We construct high-frequency data for open source:
  - since 2022;
  - contains options and futures data.

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References

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