

# Generating Multivariate Extremes via a Radial-Angular Decomposition

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Joint work with Ed Mackay, Callum Murphy-Bartrop,  
and Phil Jonathan

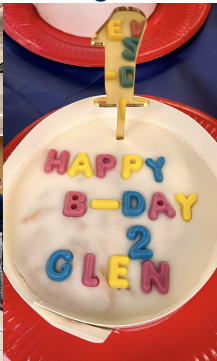
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12/06/2026



# GLE<sup>2</sup>N - The Glasgow–Edinburgh Extremes Network

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## GLEN-ECFI LECTURE

### ON RISK & EXTREMES IN ENVIRONMENTAL SCIENCES

"Predicting Multivariate Extremes in Environmental Sciences", will be presented by Philippe Naveau, research scientist at the Laboratoire des Sciences du Climat et l'Environnement.

Friday 26 June  
11 AM - 12 PM

Room 2.35  
Edinburgh Futures Institute

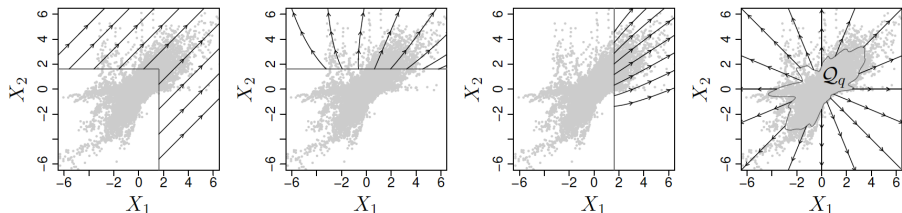
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# Multivariate extremes

Traditional models capture one type of multivariate extreme or are limited in the **strength/class** of extremal dependence they can capture.

We reproduce all joint extremes, rather than one variable being large, using a **generative model**.



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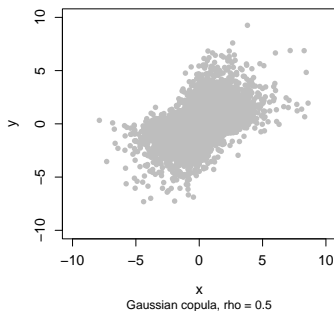
Papastathopoulos, I., De Monte, L., Campbell, R., & Rue, H. (2023). Statistical inference for radially-stable generalized Pareto distributions and return level-sets in geometric extremes. arXiv:2310.06130.

# Angular-radial representation

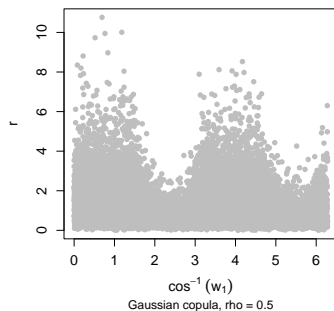
Let  $\mathbf{X} \in \mathbb{R}^d$  and

$$R := \|\mathbf{X}\| > 0, \quad \mathbf{W} := \frac{\mathbf{X}}{\|\mathbf{X}\|} \in \mathcal{S}^{d-1}.$$

Laplace margins

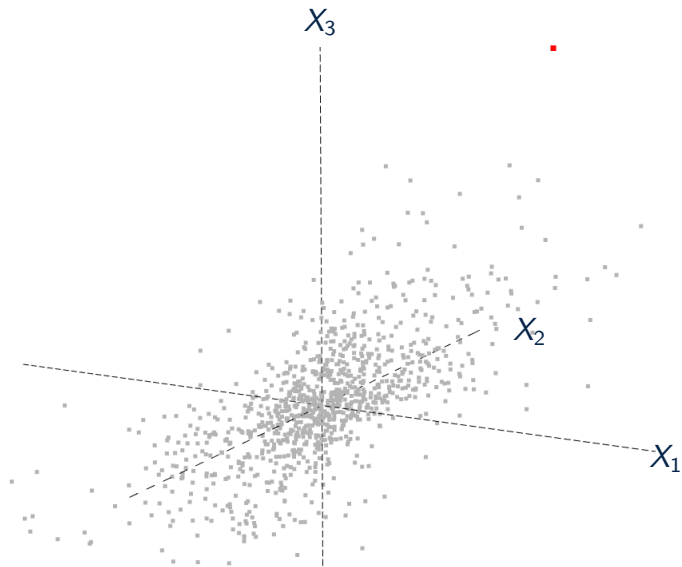


Angular-radial decomposition

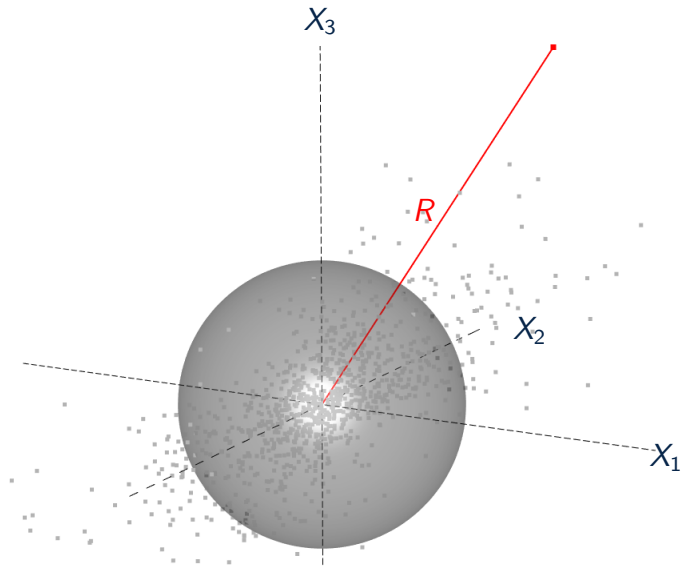


Murphy-Bartrop, C. J., Majumder, R., & Richards, J. (2024). Deep learning of multivariate extremes via a geometric representation. arXiv:2406.19936.

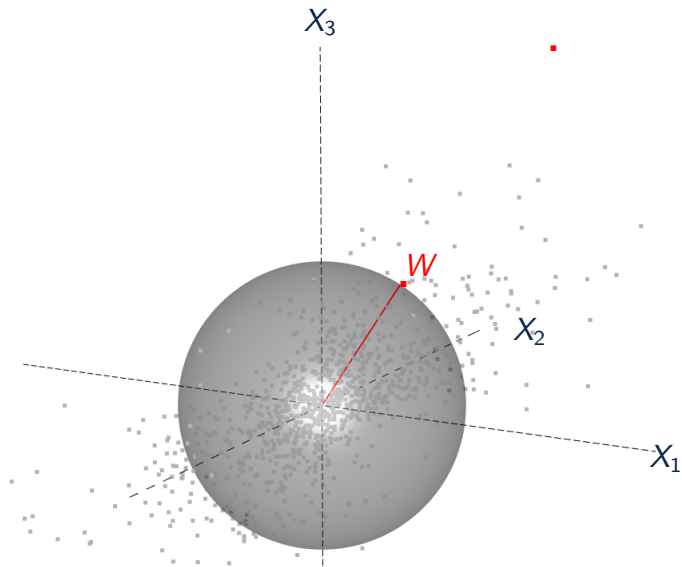
# Angular-radial representations



# Angular-radial representations



# Angular-radial representations



# Angular-radial representations

The joint density of  $(R, \mathbf{W})$  can be written as

$$f_{R, \mathbf{W}}(r, \mathbf{w}) = f_{\mathbf{W}}(\mathbf{w})f_{R|\mathbf{W}}(r|\mathbf{w}). \quad (1)$$

The Semi-Parametric Angular-Radial model (SPAR; Mackay and Jonathan, 2023) replaces  $f_{R|\mathbf{W}}(r|\mathbf{w})$  with a conditional GP model **when  $r$  is big**, i.e.,

$$f_{R|\mathbf{W}}(r|\mathbf{w}) \approx \zeta f_{\text{GP}}(r - u(\mathbf{w}); \tau(\mathbf{w}), \xi(\mathbf{w})), \quad (2)$$

for large  $r > u(\mathbf{w})$  and where  $\zeta := \Pr(R > u(\mathbf{w})|\mathbf{W} = \mathbf{w})$  is fixed.

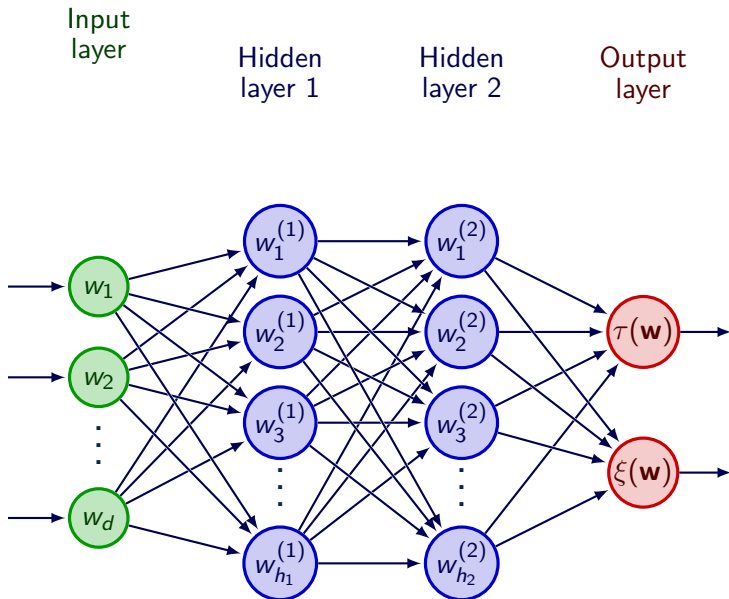
- Mackay et al. (2025) models  $(u(\mathbf{w}), \tau(\mathbf{w}), \xi(\mathbf{w}))$  via **deep extremal regression**.

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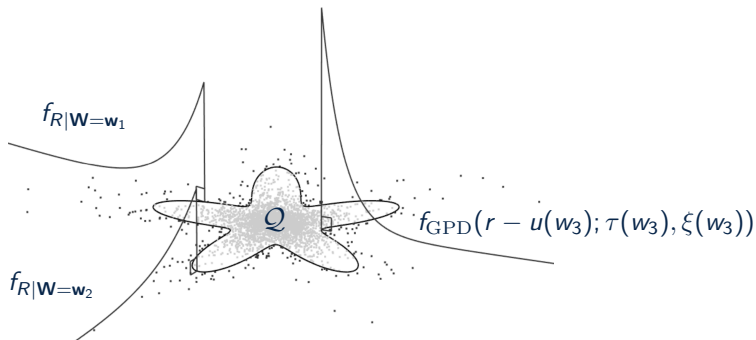
Mackay, Jonathan (2023). Modelling multivariate extremes through angular-radial decomposition of the density function. arXiv.

Mackay, Murphy-Barltrop, Richards, Jonathan (2025). Deep Learning Joint Extremes of Metocean Variables Using the SPAR Model. JOMAE.

# Deep regression for extremes



# Angular-radial representations



Quantile set  $Q := \{\mathbf{x} : \|\mathbf{x}\| < u(\mathbf{x}/\|\mathbf{x}\|)\}$  with  $\Pr\{\mathbf{X} \in Q\} = 1 - \zeta$ .

# Angular-radial representations

Recall

$$f_{R,\mathbf{W}}(r, \mathbf{w}) \approx f_{\mathbf{W}}(\mathbf{w}) \zeta f_{\text{GP}}(r - u(\mathbf{w}); \tau(\mathbf{w}), \xi(\mathbf{w})), \quad r > u(\mathbf{w}). \quad (3)$$

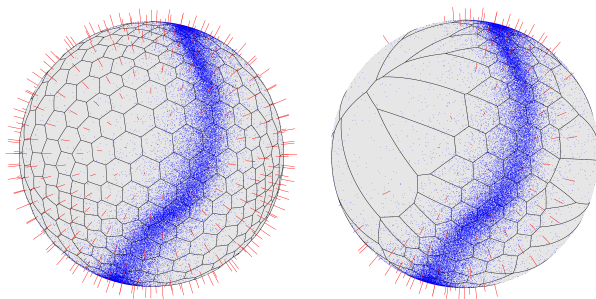
With estimates of  $(u(\mathbf{w}), \tau(\mathbf{w}), \xi(\mathbf{w}))$ , simulation from  $f_{R|\mathbf{W}}$  is simple.

In combination with a model for  $f_{\mathbf{W}}$  (see, e.g., Wessel et al., 2025), we can simulate from  $(R, \mathbf{W})$  by drawing from (3) with prob.  $\zeta$  and, otherwise, from the corresponding *non-extreme* empirical distribution.

- Both  $u(\mathbf{w})$  and  $(\tau(\mathbf{w}), \xi(\mathbf{w}))$  are represented as MLPs.
- Models fitted using variants of **stochastic gradient descent** in Keras for R;
- For GPD model, with negative log-likelihood as loss. See, e.g., Pasche and Engelke (2024) and Richards et al. (2023).
- Architecture comparison performed by brute-force  $\Rightarrow$  try different architectures, and evaluate diagnostics (QQ plots) on test data.

# Local diagnostics

Voronoi partition on angles - define non-overlapping bins  $\mathcal{B}_j$  such that  $\cap_j \mathcal{B}_j = \mathcal{S}^{d-1}$ .



# Exponential residuals

For observed data  $\{r_i, \mathbf{w}_i\}_{i=1}^n$ , define:

$$Z_i = -\log F_{\text{GP}}\{r_i - \hat{u}(\mathbf{w}_i); \hat{\tau}(\mathbf{w}_i), \hat{\xi}(\mathbf{w}_i)\}, \quad i = 1, \dots, n.$$

If the model is correct, then  $Z_i \sim \text{Exp}(1)$ ,  $i = 1, \dots, n$ . We can then diagnose model fit by looking at exponential QQ plots.

We can do this for each individual bin. To account for different numbers of points in each bin, we define standardised ordered exponential residuals

$$D_k = \mathbb{E}[Z_{(k)}] - Z_{(k)},$$

with  $Z_{(1)} > Z_{(2)} > \dots$

# Standardised Tail Plot

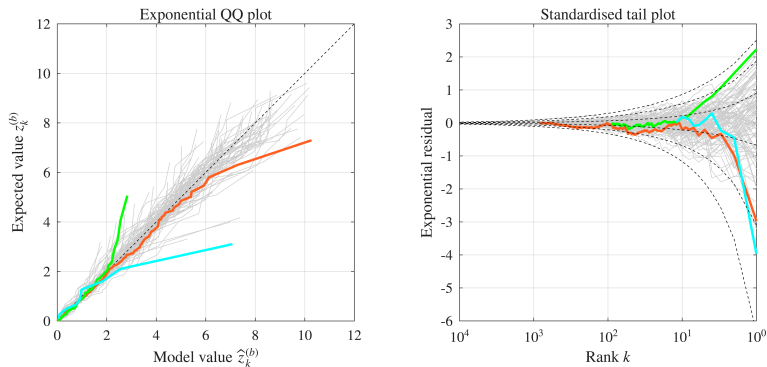


Figure: Left: Example simulations for 100 samples of random size  $N$ , where  $\log_{10}(N) \sim U(1, 4)$ . Right: Same data transformed to standardised scale, together with quantiles of the asymptotic distribution at non-exceedance probabilities 0.001, 0.025, 0.25, 0.75, 0.975, and 0.999 (dashed lines).

# Metocean multivariate extremes

- 31-year hindcast of hourly values for a site in Celtic sea ( $n \approx 270000$ ).
- $d = 5$ ,  $x$ - and  $y$ -components of wind speed ( $U$ ) and wave height ( $H$ ), and (log)-wave period ( $T_m$ ).
- Ocean storms driven by non-convex combinations of marginal “extremes” with physical limits ( $s_{\max}$ )

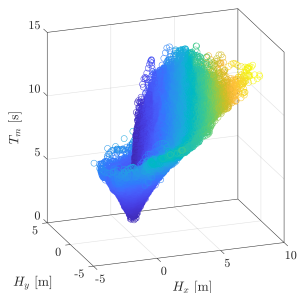
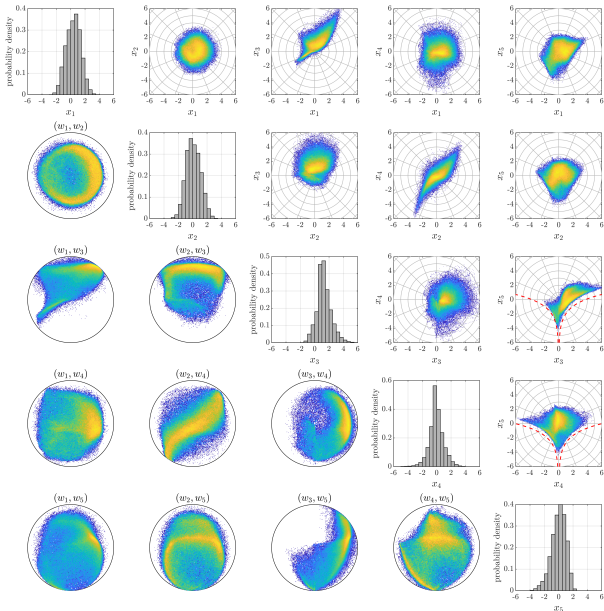


Figure:  $x$ - $y$  components of  $H_s$  and  $T_m$ . Colours indicate the value of  $H_s = \sqrt{H_x^2 + H_y^2}$  (low, high), which relates to wave steepness  $s = 2\pi H_s / (gT_m^2)$ .

# Metocean multivariate extremes



# Model

- For  $(\tau(\mathbf{w}), \xi(\mathbf{w}))$ , use MLP with 3 Layers, 16 neurons. Runs on standard laptop.
- $\zeta = 0.1$  via threshold stability plots.
- KDE for angles.

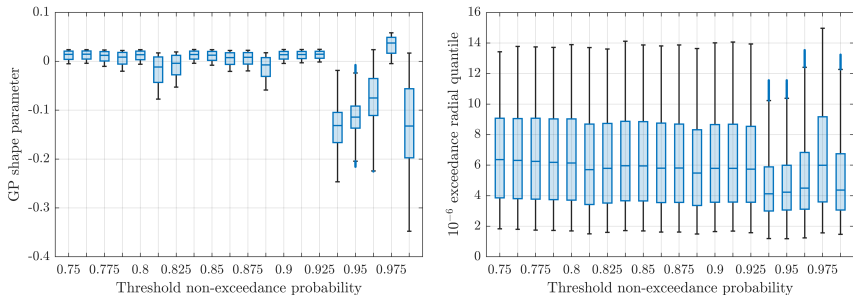
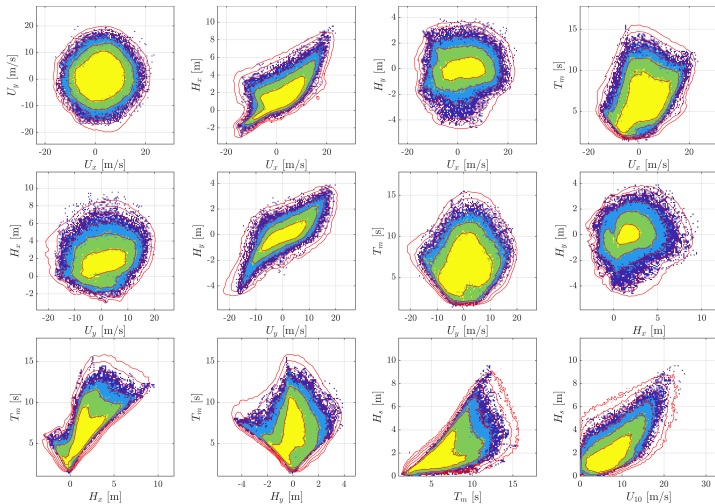


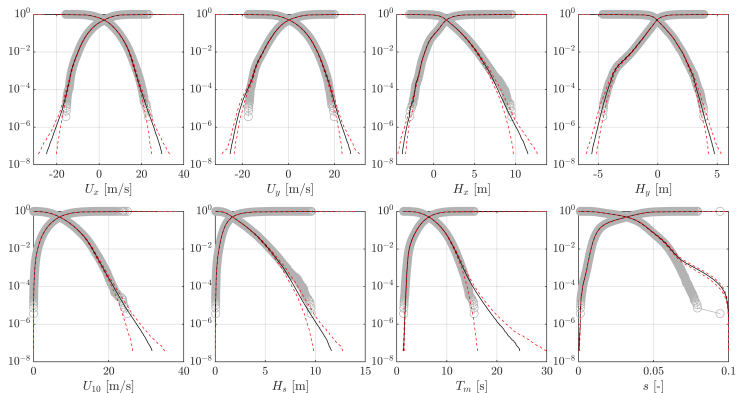
Figure: Box-plots of  $\xi(\mathbf{w})$  estimates (left) and conditional radial quantile at exceedance level  $10^{-6}$  (right).

# Estimates



Contours produced from deepSPAR model using 3000 years of reps.

# Marginal diagnostics



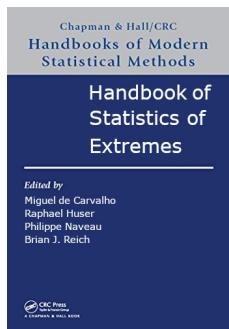
Wind speed:  $U_{10} = \sqrt{U_x^2 + U_y^2}$ . Steepness:  $s = 2\pi H_s / (gT_m^2)$ .

# Conclusion

- Fast, flexible model for generating multivariate extremes in any direction.
- No marginal model - just applied straight to observed data (subject to a choice of centre).
- New diagnostics for regression models can help with model design.
- Code available on GitHub:
  - [callumbarltrop/DeepSPAR](#);
- New work looking at temporal changes in Alpine runoff (Murphy-Barltrop et al., 2026).
- Future work; combining with a suitable model for temporal dependence.

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Murphy-Barltrop, Richards, Poschlod, Sasse, Zscheischler (2026). Exploring climate change effects on concurrent floods and concurrent droughts via statistical deep learning. arXiv.



Chapter 21. Richards and Huser (2026).

Extreme Quantile Regression with Deep Learning.

Code and short courses on GitHub - <https://github.com/Jbrich95/>

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## Bridging Heavy Tails and Artificial Intelligence

 Participating journal: [Extremes](#)



Closed for submissions

The increasing frequency and severity of extreme events—such as catastrophic flooding, record-breaking temperatures, and unprecedented heatwaves—has highlighted the urgent need for innovative approaches in risk assessment and modeling. Modern advancements in data-collection techniques have provided increasingly large and complex datasets, which can only be processed using fast and scalable algorithms and computational software. This special issue aims to bridge the gap between Artificial Intelligence (AI) and Extreme Value Theory (EVT) to harness the strengths of both fields and address the growing challenges posed by these extreme events.



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Extremes is a dedicated platform for the publication of original research in statistical extreme value theory and its applications across various fields.

Publishing model	Hybrid
Journal Impact Factor	1.1 (2023)
Downloads	58k (2024)
Submission to first decision (median)	7 days

<https://link.springer.com/collections/haghbdfdhb>

# References I

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